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On the Number of Maximum Genus Embeddings of Almost all Graphs

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A lower bound for the number of maximum genus orientable embeddings of almost all graphs is provided. This bound is sufficiently strong to demonstrate that for those complete graphs that triangulate some orientable surface, the number of maximum genus embeddings far exceeds the number of minimum genus embeddings. The proof is achieved by performing an induction in the context of permutation–partition pairs.

1. INTRODUCTION

A cellular embedding of a graph G on a closed orientable surface is one in which the regions are homeomorphic to the open unit disc. Here the graph G is permitted to have loops and multiple edges, and the regions of the embedding are the connected components of the complement of the image of G on the ambient surface. The reader is referred to [9, 16, 25] for a general introduction to this discipline. Such embeddings are governed by the Euler–Poincaré equation, which states that if the graph G is connected, then

$$p - q + r = 2 - 2g$$

where p and q denote the number of vertices and edges of G, r is the number of regions, and g is the genus of the ambient surface. The maximum genus of the graph G is the largest integer $\mu(G)$ such that G can be 2-cell embedded on the closed orientable surface of genus $\mu(G)$. This invariant has been the subject of a considerable amount of study [6, 10, 15, 26], and is considered to be well understood.

In this note we shall be mostly concerned with embeddings of the graph G in which the number of regions is either 1 or 2. It follows immediately from the Euler–Poincaré equation that these two possibilities are mutually exclusive. It also follows from the same equation that every such embedding is necessarily a maximum genus embedding. The converse is well known to be false [10, 26].

Two cellular embeddings of the same graph G an oriented surfaces are said to be equivalent if there is an orientation preserving homeomorphism of the ambient surfaces that matches up the images of the graphs on the surfaces [9]. If equivalent embeddings are regarded as being identical, then it is known [9] that the number of distinct cellular embeddings of the graph G is

$$\prod_{i=1}^{p} (d_i - 1)!$$

where $d_1, d_2, \ldots, d_p$ is the degree sequence of G.

The purpose of this note is to substantiate the feeling, common amongst topological graph theorists, that most graphs have relatively many maximum genus embeddings. For example, it was demonstrated in [24] that amongst all embeddings of all cubic pseudographs and q edges, the proportion of those with only one region is

$$\frac{1}{q} + O\left(\frac{1}{q^2}\right).$$

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It was also demonstrated in [21] that the expected number of regions in a randomly selected embedding of G is at most $p(2 + \ln \delta^*)$, where δ^* is the geometric mean of the degrees of the vertices G.

The main theorem of this note states that the number of maximum genus embeddings of almost all graphs G is not less than

$$(d_1 - 5)!\,(d_2 - 5)!\,(d_3 - 5)!\,(d_4 - 5)! \prod_{i=5}^{p} (d_i - 2)!, \tag{1}$$

where $d_1, d_2, \ldots, d_p$ is a suitable reordering of the degree sequence of G.

In particular, it follows that the complete graph K_m possesses at least $[(m-6)!]^4[(m-3)!]^{m-4}$ maximum genus embeddings. On the other hand, it is shown in Proposition 9 that the number of minimum genus embeddings of those K_m that triangulate some orientable surface is at most $(m-2)!\,(m-3)! \cdots 2!\,1!$, a much smaller quantity.

While the bound of (1) is relatively large, it is still far from being sharp. For example, it is known [23] that the number of maximum genus embeddings of the wheel with n spokes (n even) is asymptotic to $(n-2)!\,2^{n+1}$, a number that is considerably larger than the lower bound of $(n-2)!$ that is given by (1).

Results of this type come under the more general heading of genus, or region, distribution problems. There has been a considerable amount of activity in this area recently, [5–8, 11–14, 17, 20–24].

2. Permutation–Partition Pairs

The proof of Theorem 5 relies heavily on the notion of a permutation–partition pair, a generalization of graphs that has proven to be a very effective tool in resolving a variety of problems regarding the genus distribution and the genus of the amalgamation of graphs [2, 3, 18, 19]. The main points regarding these pairs are summarized below, and the reader is referred to [20] for detailed explanations and examples.

A *permutation–partition pair* is a pair (P, Π), where P is a permutation and Π is a partition, both defined over a common underlying set S the elements of which are called *bits*. If the set S contains n bits, and the partition Π consists of k non-empty sets $\Pi_1, \Pi_2, \ldots, \Pi_k$, then we refer to (P, Π) as a *$PP(n, k)$ pair*. The sets Π_i are called the *vertices* of (P, Π). If Q is a permutation of S, each of the orbits of which agrees with one of the vertices of (P, Π), we shall refer to Q as an *embedding* of (P, Π), and the orbits of the (left to right) composition PQ are called the *regions* of this embedding.

If G is a graph, and its edges are replaced by pairs of oppositely oriented arcs, then we associate with G the permutation–partition pair (P_G, Π_G), where the underlying set S_G consists of all these arcs, the permutation P_G is the fixed point free involution that maps each arc to its opposite, and each member of Π_G consists of all the arcs emanating from one of the vertices of G. Such pairs are called *graphical* pairs, and it is clear that a pair (P, Π) is graphical if P is an involution without fixed points.

There is a one-to-one correspondence between the orientable cellular embeddings of G and the embeddings of the pair (P_G, Π_G). This correspondence matches up the boundaries of the geometric regions of G with the regions of the associated pair (P_G, Π_G). We shall be mainly concerned with embeddings of (P, Π) that have 1 or 2 regions. A straightforward parity argument shows that, for a given pair, these two cases are mutually exclusive.

If b is any bit in S, then we denote by Π/b the partition of $S - \{b\}$ that is obtained by simply deleting b from the member of Π that contains it. Similarly, P/b denotes the permutation obtained by deleting the occurrence of b in the disjoint cycle decomposition of P. If a and b are two (not necessarily distinct) bits that belong to the same member of Π, we write $a \equiv b(\bmod \Pi)$. If $a \equiv b(\bmod \Pi)$ then the pair $(P, \Pi)/a \to b$ is the pair (P', Π'), where

$$\Pi' = \Pi/b,$$
$$P' = \begin{cases} P(b\,a\,bP)/b & \text{if} \quad a,\ b \text{ and } bP \text{ are all distinct} \\ P(b\,a)/b = P/b & \text{if} \quad bP = a \neq b \\ P/b & \text{if} \quad bP = b \neq a \\ P/b & \text{if} \quad a = b \text{ and } \{b\} \in \Pi. \end{cases}$$

The following lemmas will prove crucial in the proofs below, and their validity follows immediately from the above definition.

Lemma 1. *Let (P, Π) be a permutation–partition pair, and let a and b be two bits such that $a \equiv b(\bmod \Pi)$. If $(P', \Pi') = (P, \Pi)/a \to b$, then $xP' = xP$ whenever $xP \notin \{b, a, bP\}$. Moreover, if $b = bP$, then $xP' = xP$ for all $x \neq b$.*

Lemma 2. *Let (P, Π) be a permutation–partition pair, and let a, b, x be bits and Π_i a member of Π such that*

$$\Pi_i \supset \{a, b\} \qquad \text{and} \qquad \Pi_i \cap \{x, xP\} = \varnothing.$$

Then $xP' = xP$, where $(P', \Pi') = (P, \Pi)/a \to b$.

The function $r_{(P,\Pi)}(k)$ denotes the number of embeddings of (P, Π) that have k regions. A simple argument on the parity of permutations implies that if $r_{(P,\Pi)}(k) \neq 0$ for some k, then $r_{(P,\Pi)}(k \pm 1) = 0$.

We now restate Corollaries 1.2 and 1.4 of [20] as the following two lemmas. In both lemmas $\delta_{x,y}$ denotes the Kronecker delta function, which is 0 or 1 according as to whether or not x and y are distinct.

Lemma 3. *If (P, Π) is a permutation–partition pair and b is a bit such that $\{b\} \notin \Pi$, then*

$$r_{(P,\Pi)}(k) = \sum_{\substack{a \equiv b(\bmod \Pi) \\ a \neq b}} r_{(P,\Pi)/a \to b}(k - \delta_{a,bP}).$$

Lemma 4. *If (P, Π) is a permutation–partition pair and b is a bit such that $\{b\} \in \Pi$, then*

$$r_{(P,\Pi)}(k) = r_{(P,\Pi)/b \to b}(k - \delta_{b,bP}).$$

3. The Main Theorem

A permutation–partition pair (P, Π) with $\Pi = \{\Pi_i \mid i = 1, 2, \ldots, k\}$ and $d_i = |\Pi_i|$ for all such i, is said to be *ample* provided that the number of its embeddings with at most two regions is at least

$$(d_1, d_2, \ldots, d_k)!_{-2} = (d_1 - 5)!\,(d_2 - 5)!\,(d_3 - 5)!\,(d_4 - 5)!\,(d_5 - 2)! \cdots (d_k - 2)!,$$

where $m! = 1$ whenever m is a non-positive integer.

A $PP(n, k)$ pair (P, Π) is said to be *strongly K_4 based* if $k \geq 4$ and the constituent members of the partition Π can be reindexed so that:

(1) for each i, $5 \leq i \leq k$, there exist distinct bits $x_i, y_i \in \Pi_i$ such that $x_iP \in \Pi_s$ and $y_iP \in \Pi_t$ for some $s, t < i$;

(2) there exist bits $a_i, b_i, c_i \in \Pi_i$, $i = 1, 2, 3, 4$ such that the disjoint cycle decomposition of P contains the factor

$$(a_1\, c_2)(b_1\, b_3)(c_1\, a_4)(b_2\, b_4)(a_2\, c_3)(a_3\, c_4).$$

If the pair (P, Π) corresponds to a graph G, then the pair is strongly K_4-based whenever the vertices of the graph can be ordered $v_1, v_2, \ldots, v_k$, so that v_1, v_2, v_3, v_4 span a copy of K_4 in G and each vertex v_i, $i > 4$, is joined by at least 2 distinct edges to the set $\{v_1, v_2, \ldots, v_{i-1}\}$. Condition 1 makes the induction step possible, and condition 2 anchors the process.

THEOREM 5. *Every strongly K_4-based permutation–partition pair is ample.*

PROOF. We first show, by a minimum counterexample argument, that all strongly K_4-based $PP(n, 4)$ pairs in which $d_1, d_2, d_3, d_4 \leq 5$ are ample. Note that for such pairs being ample means simply having at least one embedding with at most 2 regions. It will be convenient to refer to these pairs as the *small* pairs, and we now make three observations that dispose of most of these small pairs.

(1) If (P, Π) is a graphical small pair, say $(P, \Pi) = (P_G, \Pi_G)$, then condition 2 of the definition of strongly K_4-based pairs means that G contains a copy of K_4. Consequently, since G has only 4 vertices, it follows that G has a spanning path the complement of which in G is connected, and so, by the Jungermann–Xuong Theorem [10, 26], G is upper embeddable. In other words, (P, Π) has an embedding with at most two regions.

(2) Let (P, Π) be a small pair with $b \in \Pi_i$ a fixed point of P. It follows that the smaller pair $(P, \Pi)/a_i \to b$ is obtained from (P, Π) by simply suppressing the occurrences of b in both P and Π; and so, by Lemma 1, it is also strongly K_4-based. Moreover, since $\delta_{a_i, bP}$ is 0 it follows from Lemma 3 that (P, Π) is ample whenever the smaller pair $(P, \Pi)/a_i \to b$ is ample.

(3) Let (P, Π) be a small pair with distinct bits $a, b \in \Pi_i$ such that a, b are both distinct from a_i, b_i and c_i. Suppose further that $bP \neq a$. It then follows from Lemma 1 that the smaller pair $(P, \Pi)/a \to b$ is also strongly K_4-based and it follows from Lemma 3 that (P, Π) is ample whenever $(P, \Pi)/a \to b$ is ample.

Of all the small strongly K_4-based $PP(n, 4)$ pairs that are not ample, let (P, Π) be one with the smallest possible value of n. It follows from the above observations that (P, Π) cannot be graphical, and P has no fixed points, and that if $a, b \in \Pi_i$ such that a, b are distinct from a_i, b_i, and c_i, then $bP = a$ and $aP = b$. In view of the fact that each d_i is at most 5, this means that the disjoint cycle decomposition of P must contain exactly one cycle of length 3 or 4. Up to symmetry, there are only 3 such pairs possible. These are listed below, each together with an embedding Q that bears witness to the ampleness of the pair:

(a)
$$P = (a_1\, c_2)(b_1\, b_3)(c_1\, a_4)(b_2\, b_4)(a_2\, c_3)(a_3\, c_4)(1\,2\,3),$$
$$\Pi = \{1, a_1, b_1, c_1 \mid 2, a_2, b_2, c_2 \mid 3, a_3, b_3, c_3 \mid a_4, b_4, c_4\},$$
$$Q = (a_1\, b_1\, c_1\, 1)(a_2\, 2\, c_2\, b_2)(a_3\, b_3\, c_3\, 3)(a_4\, b_4\, c_4);$$

(b)
$$P = (a_1\, c_2)(b_1\, b_3)(c_1\, a_4)(b_2\, b_4)(a_2\, c_3)(a_3\, c_4)(1\,2\,3\,4),$$
$$\Pi = \{1, a_1, b_1, c_1 \mid 2, a_2, b_2, c_2 \mid 3, a_3, b_3, c_3 \mid 4, a_4, b_4, c_4\},$$
$$Q = (a_1\, b_1\, c_1\, 1)(a_2\, 2\, c_2\, b_2)(a_3\, b_3\, c_3\, 3)(a_4\, 4\, c_4\, b_4);$$

(c)
$$P = (a_1\, c_2)(b_1\, b_3)(c_1\, a_4)(b_2\, b_4)(a_2\, c_3)(a_3 c_4)(1\,2\,3)(4\,5),$$
$$\Pi = \{1, a_1, b_1, c_1 \mid 2, a_2, b_2, c_2 \mid 3, a_3, b_3, c_3 \mid 4, 5, a_4, b_4, c_4\},$$
$$Q = (a_1\, b_1\, c_1\, 1)(a_2\, 2\, c_2\, b_2)(a_3\, b_3\, c_3\, 3)(a_4\, 4\, c_4\, 5\, b_4).$$

This shows that all the small strongly K_4-based pairs are indeed ample.

We now proceed by a double induction, first on k and then on d_k. If $k = 4$, then a symmetry argument allows us to assume that $d_1 \leqslant d_2 \leqslant d_3 \leqslant d_4$. Let $D \geqslant 6$ be a positive integer and assume that the theorem holds for all $PP(n, 4)$ pairs such that $d_4 < D$. Let (P, Π) be a $PP(n, 4)$ pair with $d_4 = D$. Let b be a bit of Π_4 that is distinct from a_4, b_4 and c_4. Since $a_4 = c_1P$, $b_4 = c_2P$ and $c_4 = a_3P$, it follows that bP is also distinct from a_4, b_4 and c_4. Note that there exist at least $D - 5 \geqslant 1$ bits $a \in \Pi_4$ such that a is distinct from b, bP, a_4, b_4 and c_4. For these bits, the following hold:

$$b \text{ and } bP \quad \text{are both distinct from} \quad a \tag{2}$$

$$a_4P,\, b_4P,\, c_4P \quad \text{are all distinct from} \quad b,\, a,\, bP \tag{3}$$

For each such bit a set

$$(P^a, \Pi') = (P, \Pi)/a \to b.$$

It follows from (2) above that $\delta_{a,bP} = 0$ and it follows from (3) and Lemmas 1 and 2 that $xP = xP^a$ for all $x \in \{a_i, b_i, c_i \mid i = 1, 2, 3, 4\}$. Thus each such pair (P^a, Π') is also strongly K_4-based, and hence, by the induction hypothesis, it is also ample. By Lemma 3 the number of embeddings of (P, Π) that have at most two regions is therefore at least

$$(D-5)[(d_1-5)!\,(d_2-5)!\,(d_3-5)!\,(D-1-5)!] = (d_1-5)!\,(d_2-5)!\,(d_3-5)!\,(d_4-5)!.$$

This establishes the theorem for all strongly K_4-based $PP(n, 4)$ pairs.

Let $K \geqslant 5$ be an integer and assume that the theorem has been established for all $PP(n, k)$ pairs with $k < K$. Let (P, Π) be a strongly K_4-based $PP(n, K)$ pair. Since Π_K contains the distinct bits x_K and y_K, it follows that $d_K \geqslant 2$. If $d_K = 2$, set

$$(P', \Pi') = (P, \Pi)/x_K \to y_K \qquad \text{and} \qquad (P'', \Pi'') = (P', \Pi')/x_K \to x_K.$$

Note that $y_KP \neq x_K$ and $x_KP' = y_KP \neq x_K$ and hence the Kronecker deltas that are involved in both these reductions are 0. Moreover, the pair (P'', Π''), which, by Lemmas 1 and 2, inherits the property of being strongly K_4-based from (P, Π), is a $PP(n, K-1)$ pair. Hence, by the induction hypothesis and Lemmas 3 and 4, the number of minimum embeddings of (P, Π) is at least

$$(d_1, d_2, \ldots, d_{K-1})!_{-2} = (d_1, d_2, \ldots, d_K)!_{-2}$$

Finally, let $D \geqslant 3$ be an integer and assume that the theorem holds for all $PP(n, K)$ pairs for which $p_K < D$. Let (P, Π) be a $PP(n, K)$ pair with $d_K = D$. Let b be a bit of Π_K that is distinct from both x_K and y_K. If a is any bit of Π_K that is distinct from both b and bP, let

$$(P^a, \Pi') = (P, \Pi)/a \to b.$$

There are clearly at least $D - 2$ such bits a, and for each of them

$$\{x_KP, y_KP\} \cap \{b, a, bP\} = \varnothing,$$

so that, by Lemma 1, $x_KP^a = x_KP$ and $y_KP^a = y_K$. Thus each of these pairs (P^a, Π') is again strongly K_4-based, and for each of them $|\Pi'_K| = D - 1$. Since $a \neq bP$, $\delta_{a,bP} = 0$. Hence, by the induction hypothesis, the number of embeddings of the given pair (P, Π) that have at most 2 regions is at least

$$(D-2)(d_1, d_2, \ldots, d_{K-1}, D-1)!_{-2} = (d_1, d_2, \ldots, d_K)!_{-2}.$$

This completes both the induction process and the proof of the theorem. □

4. Applications to Graphs

A graph G is said to be *strongly K_4-based* if its associated permutation–partition is strongly K_4-based. This means that the vertices of G can be labelled $v_1, v_2, \ldots, v_k$ so that:

(1) for each i, $5 \leq i \leq k$, the vertex v_i is joined to the set $\{v_1, v_2, \ldots, v_{i-1}\}$ by at least 2 edges;

(2) v_1, v_2, v_3, v_4 span a subgraph of G that is isomorphic to K_4.

Corollary 6. *Every strongly K_4-based graph is ample.*

Proof. This follows immediately from Theorem 5. □

Trees provide the counterexample needed to show that not all graphs are ample.

Corollary 7. *The complete graph on m vertices has at least $[(m-6)!]^4(m-3)!]^{m-4}$ maximum genus embeddings.*

The *random simple graph on k vertices* is a simple graph on k labelled vertices such that every 2 distinct vertices are joined with a probability of $\frac{1}{2}$. A property P of graphs is said to hold for *almost all graphs* if the probability of its holding for the random simple graph on k vertices converges to 1 as k becomes indefinitely large. The reader is referred to [4] for an in-depth treatment of this topic.

Theorem 8. *Almost all graphs are ample.*

Proof. It clearly suffices to show that the probability that the random simple graph on k vertices is strongly K_4-based converges to 1 as $k \to \infty$. The probability $Q(k)$ that the random simple graph G on k vertices is strongly K_4-based is at least

$$Q(k) = q(k)q_4(k)q_5(k) \cdots q_{k-1}(k),$$

where $q(k)$ is the probability that G contains a copy of K_4, and for each $4 \leq i \leq k-1$, $q_i(k)$ denotes the probability that, given a set U_i of i vertices of G, there is a vertex of $V(G) - U_i$ that is adjacent to at least 2 vertices of U_i.

Given any 4 vertices of G, the probability that they span a copy of K_4 is $\frac{1}{64} > 0$, and so $q(k)$ clearly converges to 1.

Turning to $q_i(k)$ for $i = 4, 5, \ldots, k-1$, we note that

$$q_i(k) = 1 - \left[\frac{1}{2^i} + \frac{i}{2^i}\right]^{k-i}.$$

Hence,

$$\begin{aligned}\prod_{i=4}^{k-1} q_i(k) &\geq \prod_{i=4}^{k-1} \left\{1 - \left[\frac{i+1}{2^i}\right]^{k-i}\right\} \\ &\geq 1 - \sum_{i=4}^{k-1} \left[\frac{i+1}{2^i}\right]^{k-i} \\ &\geq 1 - \sum_{i=4}^{k-1} 2^{-(i-1)(k-i)/2} \\ &\geq 1 - \frac{k}{2^{k/2}}.\end{aligned}$$

It is therefore clear that $Q(k)$ does indeed converge to 1. □

PROPOSITION 9. *The number of triangulations by the complete graph K_m is no greater than* $(m-2)!\,(m-3)!\cdots 2!\,1!$.

PROOF. It is convenient to present this proof in the older terminology of rotation systems. Let $v_1, v_2, \ldots, v_m$ denote the vertices of the complete graph K_m, and let $e_{i,j}$ denote the arc from v_i to v_j. Note that, for any triangulation, once it has been specified that in the local rotation at the vertex v_i the arc $e_{i,j}$ is followed by the arc $e_{i,k}$, then we can conclude that $e_{j,k}$ is followed by $e_{j,i}$ near v_j, and that $e_{k,i}$ is followed by $e_{k,j}$ at v_k.

There are clearly $(m-2)!$ choices for the local rotation at v_1. In view of the above observation, once such a choice has been made, there are at most $(m-3)!$ consistent choices for the local rotation at v_2. These choices for the local rotations at v_1 and v_2 leave room for at most $(m-4)!$ choices at v_3, and so on. □

Since we know by the Ringel–Youngs Theorem [16] that such triangulations do indeed yield the minimum genus for K_m for $m \equiv 0, 3, 4, 7 \pmod{12}$, it follows that for these values of m, K_m does indeed have many more maximum genus embeddings than minimum genus embeddings.

5. AN ALTERNATIVE APPROACH

Both the statement and the proof of Theorem 5 are made more complicated than absolutely necessary by the wish to apply them to simple graphs. Alternatively, one can define a pair (P, Π) to be *strongly based* provided that the constituent members of Π can be reindexed so that:

for each i, $2 \leq i \leq k$, there exist distinct bits $x_i, y_i \in \Pi_i$ such that $x_iP \in \Pi_s$ and $y_iP \in \Pi_t$ for some $s, t < i$.

A considerable simplification of the proof of Theorem 5 above yields the following alternative version.

THEOREM 5′. *Every strongly based $PP(n, k)$ pair has at least*

$$\prod_{i=1}^{k} (d_i - 2)!$$

embeddings with at most 2 *regions.*

Theorem II.3 of [22] shows that this lower bound is asymptotically sharp for the bouquets with an even number of loops.

A graph is said to be *strongly based* when its vertices can be ordered as $v_1, v_2, \ldots, v_p$ so that, for any $i > 1$, v_i is joined by 2 distinct edges to $\{v_1, v_2, \ldots, v_{i-1}\}$; i.e., whenever its associated pair (P_G, Π_G) is strongly based, then one obtains the following result from Theorem 5′.

COROLLARY 6′. *Every strongly based graph has at least*

$$\prod_{i=1}^{k} (d_i - 2)!$$

maximum genus embeddings.

Unfortunately, the condition that v_2 be adjacent twice to v_1 precludes all simple graphs from being strongly based. However, one can still sharpen the lower bound of

Corollary 7 somewhat, as follows. Let $K_m + e$ denote the multigraph obtained by doubling one of the edges of the complete graph K_m. The graph $K_m + e$ is clearly strongly based, and so, by Corollary 6′, it has at least $[(m-2)!]^2[(m-3)!]^{m-2}$ maximum genus embeddings. For $m \equiv 0, 3 \pmod 4$ every maximum genus embedding of $K_m + e$ has one region. If the extra edge is now deleted, a 2-region embedding of K_m is obtained, which is of course also a maximum genus embedding. Since every embedding of K_m is obtained from $(m-1)^2$ embeddings of K_m by this deletion process, it follows that for $m \equiv 0, 3 \pmod 4$ K_m has at least

$$\left(\frac{m-2}{m-1}\right)^2[(m-3)!]^m$$

maximum genus embeddings, a result that the author believes to be still far from sharp.

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